Minimum-Time Pointing Control of a Two-Link Manipulator

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In this paper, we investigate minimum-time pointing control for the end effector of a planar, two-link manipulator. Minimum-time pointing control is a new area of research for multilink manipulators and can be applied to rapid retargeting control of a multibody spacecraft. Many researchers have investigated the minimum-time control problems of industrial robots; their results are limited, however, to moving the end effector of the manipulator to reach a particular point, to travel a given distance, or to follow a predetermined path. We consider the minimum-time control problem of aligning the second link of the two-link manipulator with a given target point. A numerical method, called the minimizing-boundary-condition method, is used to determine optimal solutions for the two-point boundary-value problem associated with first-order necessary conditions. Minimum-time solutions for different models of pointing systems are compared. The results show that a two-link manipulator with two degrees of freedom performs better than a conventional one-link system for minimum-time pointing.

I. Introduction

THE minimum-time control problem of manipulators is not new to researchers in robotics. Two subclasses of minimum-time control problems (trajectory and path planning) have been studied extensively since the paper by Kahn and Roth[1] appeared in 1971. However, the minimum-time pointing control problem has not been considered.

The objective of the minimum-time trajectory planning problem is to move the end effector of the manipulator along a prespecified path, which is usually designed to avoid obstacles, as fast as possible without violating the torque limits. For this class of problems, the joint angles are related to a single path parameter so that the number of states and controls are reduced. Numerical search procedures were used to determine the maximum velocity at each path point,[2,3] and the switching points were also determined along the path.[3] In Ref. 4, the geometrical interpretation of the torque or force constraints was used to find the maximum velocity. A good review of the work in this field can be found in Ref. 5.

Several other researchers have addressed the path planning problem. This class of problems is considered more difficult than the trajectory planning problem since the joint angles cannot be related to a single path parameter. Dynamic equations of manipulators were linearized in Ref. 1 to obtain a suboptimal feedback control, which approximated the open-loop optimal control. In Refs. 6–9, the minimum-time control problems of moving the end effector to travel a given distance or to reach a particular point were solved without using simplified dynamic equations. An extension of the gradient method based on the use of an adjustable control-variation weight matrix was used by Weinreb and Bryson[6] to obtain near-bang-bang solutions. However, the sharp discontinuities in bang-bang controls were not obtained in the direct optimization of Ref. 6. Meier and Bryson[7] used a gradient method and a switch time optimization program to search for exact bang-bang solutions in which the singular controls were approximated by high-frequency bang-bang controls. The two-point boundary-value problems associated with the first-order necessary conditions of the minimum-time control problem were solved in Refs. 8 and 9. Geering et al.[8] used parameter optimization to obtain an initial guess and then a shooting method to obtain bang-bang and singular controls for different initial conditions. Oberle[9] used a multiple-point shooting method to obtain bang-bang and singular controls without using parameter optimization. Because of the model difference, in Ref. 9 the singular control appeared at the shoulder of the manipulator whereas in Ref. 8 the singular control appeared at the elbow.

We consider the minimum-time pointing control problem of a planar, two-link manipulator in which the second link of the manipulator is aligned with a target point. The pointing control problem of multiple-link manipulators is a new area of research that can be applied to accurate aiming of an industrial robot or to rapid retargeting of a multibody spacecraft. The minimizing-boundary-condition method, developed by Chuang and Speyer,[10] is used to solve the two-point boundary-value problem associated with the first-order necessary conditions for optimality. Optimal solutions for four different pointing system models are then determined and compared for a shorter maneuvering time.

The remainder of this paper is organized as follows: The dynamic equations for the two-link manipulator without tip mass and the related minimum-time optimal control problem are presented in Sec. II. First-order necessary conditions and the associated two-point boundary-value problem are derived in Sec. III. Section IV derives the minimizing-boundary-condition method for solving the two-point boundary-value problems of Sec. III. Numerical results are presented in Sec. V, where four different pointing system models are compared. Section VI concludes the paper.

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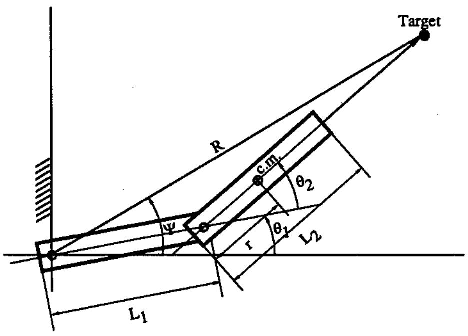

Fig. 1 A two-link pointing manipulator.

II. Problem Formulation

A planar manipulator with two uniform rigid links for a pointing system is shown in Fig. 1, where the shoulder joint is fixed to a base, L_1 is the length of the first link, L_2 the length of the second link, r the distance between the center of mass of the second link and the elbow axis, θ_1 the shoulder angle, θ_2 the elbow angle (which is the relative angular rotation of the second link with respect to the first link), Ψ the target angle, and R the target distance nondimensionalized by L_1. Four pointing system models considered in this paper are shown in Fig. 2 and described as follows:

Model A: The elbow joint is locked ($\theta_2 \equiv 0$), and the joint torque T_1 is applied at the shoulder joint.

Model B: The shoulder joint is locked ($\theta_1 \equiv 0$), and the joint torque T_2 is applied at the elbow joint.

Model C: The shoulder and elbow joints are unlocked and the joint torques T_1 and T_2 are applied at the shoulder and elbow joints, respectively.

Model D: The shoulder and elbow joints are unlocked, the joint torque T_1 is applied at the shoulder joint, and the direct torque T_2 (e.g., a reaction wheel) is applied at the second link.

Equal bounds of magnitude are assumed for the torques T_1 and T_2:

$$|T_i| \leq T_{\max}, \qquad i = 1,2 \tag{1}$$

Note that the maximum torque required to lock the shoulder joint for model B is larger than $T_{\max}$, as will be seen in Sec. V.

Define the state variables

$$x_1(t) = \theta_1(t), \qquad x_2(t) = \dot{\theta}_1(t)$$

$$x_3(t) = \theta_2(t), \qquad x_4(t) = \dot{\theta}_2(t) \tag{2}$$

and nondimensionalized control variables

$$u_i(t) = \frac{T_i(t)}{T_{\max}}, \qquad i = 1,2 \tag{3}$$

with the control bounds set to $+1$ and -1:

$$|u_i(t)| \leq 1, \qquad i = 1,2 \tag{4}$$

We write the dynamic equations as

$$\dot{x}_1 = x_2 \tag{5}$$

$$\dot{x}_2 = [I_2 u_1 - (\alpha I_2 + I_4 \cos x_3)u_2 + I_2 I_4 (x_2 + x_4)^2 \sin x_3 + I_4^2 x_2^2 \sin x_3 \cos x_3]/[I_2 I_3 - I_4^2 \cos^2 x_3] \tag{6}$$

$$\dot{x}_3 = x_4 \tag{7}$$

$$\dot{x}_4 = [-(I_2 + I_4 \cos x_3)u_1 + (\alpha I_2 + I_3 + (1+\alpha) I_4 \cos x_3)u_2 - I_4(I_3 + I_4 \cos x_3)x_2^2 \sin x_3 - I_4(I_2 + I_4 \cos x_3) \times (x_2 + x_4)^2 \sin x_3]/[I_2 I_3 - I_4^2 \cos^2 x_3] \tag{8}$$

where

α = a system parameter used to denote the type of torque applied to the second link ($\alpha = 1$ for a joint torque and $\alpha = 0$ for a direct torque)
I_1 = the mass moment of inertia of the first link with respect to the shoulder axis scaled by $T_{\max}$
I_2 = the mass moment of inertia of the second link with respect to the elbow axis scaled by $T_{\max}$
I_3 = $I_1 + m_2 L_1^2 / T_{\max}$, where m_2 is the mass of the second link
I_4 = $m_2 r L_1 / T_{\max}$

With the selection in which the two uniform links are identical, $r = L_2/2$, $I_1 = 0.25$, and $R = 100$, the other system constants are calculated to be $I_2 = 0.25$, $I_3 = 1$, and $I_4 = 0.375$. Equations (5–8) are applicable to the four models considered in this paper. By letting $x_3(t) = \dot{x}_3(t) = \ddot{x}_3(t) = \dot{x}_4(t) = 0$ and $\alpha = 1$, two first-order dynamic equations for model A follow from Eqs. (5) and (6). By letting $x_1(t) = \dot{x}_1(t) = \ddot{x}_1(t) = \dot{x}_2(t) = 0$ and $\alpha = 0$, two first-order dynamic equations for model B are derived from Eqs. (7) and (8). A multibody space pointing system similar to model B has been studied in Ref. 11; however, in Ref. 11, the mass properties of the two links are different, and the second link is flexible. Equations (5–8) represent the dynamic equations for model C when $\alpha = 1$ and for model D when $\alpha = 0$.

For a fixed target with an angle Ψ and distance R, the minimum-time pointing control problem is to find control histories to minimize a maneuvering time

$$J = \int_0^{t_f} \mathrm{d}t = t_f \tag{9}$$

subject to the dynamic constraints (5–8), the control constraint (4), and other boundary conditions. In this paper, initial conditions are assumed to be

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0 \tag{10}$$

and final conditions are assumed to be

$$x_2(t_f) = x_4(t_f) = 0 \tag{11}$$

A special final condition, which constrains the second link of the manipulator to be aligned with the target point, must also be satisfied:

$$\sin x_3(t_f)/R + \sin[\Psi - x_1(t_f) - x_3(t_f)] = 0 \tag{12}$$

where R is the target distance nondimensionalized by L_1 (see Fig. 1). Note that since the shoulder and elbow angles are not prescribed at final time, there are an infinite number of solutions that satisfy condition (12). Therefore, the admissible control space for the optimal control problem considered here is larger than those considered in Refs. 6–9.

III. First-Order Necessary Conditions

First-order necessary conditions for determining the optimal solutions for the minimum-time control problem stated in

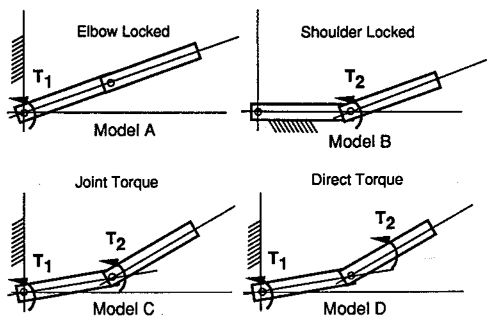

Fig. 2 Four pointing system models.

Sec. II and the associated two-point boundary-value problem with switching conditions are derived in this section.

Adjoining the dynamic equations (5-8) to the integrand of the cost function (9) with the Lagrange multiplier forms a variational Hamiltonian function

$$H(x,u,\lambda) = 1 + \lambda_1 x_2 + \lambda_2\{[I_2 u_1 - (\alpha I_2 + I_4 \cos x_3)u_2 + I_2 I_4 (x_2 + x_4)^2 \sin x_3 + I_4^2 x_2^2 \sin x_3 \cos x_3]/[I_2 I_3 - I_4^2 \cos^2 x_3]\} + \lambda_3 x_4 + \lambda_4\{[-(I_2 + I_4 \cos x_3)u_1 + (\alpha I_2 + I_3 + (1+\alpha) I_4 \cos x_3)u_2 - I_4(I_3 + I_4 \cos x_3)x_2^2 \sin x_3 - I_4(I_2 + I_4 \cos x_3)(x_2 + x_4)^2 \sin x_3]/[I_2 I_3 - I_4^2 \cos^2 x_3]\} \quad (13)$$

where $x(t) = [x_1(t),x_2(t),x_3(t),x_4(t)]^T$, $u(t) = [u_1(t),u_2(t)]^T$, and the Lagrange multiplier $\lambda(t) = [\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t)]^T$ satisfies

$$\dot{\lambda} = -\left[\frac{\partial H}{\partial x}\right]^T \quad (14)$$

Since the system is autonomous, the Hamiltonian function (13) is a constant of motion. Furthermore, since u_1 and u_2 linearly appear in the Hamiltonian function (13), the optimal controls are determined by using Pontryagin's minimum principle

$$\min_{u \in U} H \quad (15)$$

where U is the admissible control set of piecewise continuous functions constrained by Eq. (4). Condition (15) leads to the use of two switching functions

$$H_{u_1} = \lambda_2 I_2 - \lambda_4(I_2 + I_4 \cos x_3) \quad (16)$$

$$H_{u_2} = -\lambda_2(\alpha I_2 + I_4 \cos x_3) + \lambda_4(\alpha I_2 + I_3 + (1+\alpha) I_4 \cos x_3) \quad (17)$$

to determine the optimal control functions u_1 and u_2:

$$u_i(t) = \begin{cases} +1 & \text{if} \quad H_{u_i}(t) < 0 \\ -1 & \text{if} \quad H_{u_i}(t) > 0 \end{cases} \quad i = 1,2 \quad (18)$$

where the denominator of Eq. (13), $I_2 I_3 - I_4^2 \cos^2 x_3$, is eliminated from Eqs. (16) and (17) since it is a positive term. At the switching time t_s, where $H_{u_1}(t_s) = 0$ or $H_{u_2}(t_s) = 0$, the derivatives of H_{u_1} and H_{u_2} given by

$$\dot{H}_{u_1} = -\lambda_1 I_2 - [\lambda_2 I_4 \cos x_3 - \lambda_4(I_3 + I_4 \cos x_3)]\, 2 I_2 I_4 x_2 \sin x_3 / [I_2 I_4 - I_4^2 \cos^2 x_3] + \lambda_3(I_2 + I_4 \cos x_3) + \lambda_4 I_4 x_4 \sin x_3 \quad (19)$$

$$\dot{H}_{u_2} = \{\lambda_1 + [H_{u_1}(x_2 + x_4) + [\lambda_2 I_4 \cos x_3 - \lambda_4(I_3 + I_4 \cos x_3)]x_2]\, 2 I_4 \sin x_3/[I_2 I_4 - I_4^2 \cos^2 x_3]\} \times (\alpha I_2 + I_4 \cos x_3) + \lambda_2 I_4 x_4 \sin x_3 - [\lambda_3 + 2 H_{u_1} I_4 \times (x_2 + x_4) \sin x_3/[I_2 I_4 - I_4^2 \cos^2 x_3]] \times [\alpha I_2 + I_3 + (1+\alpha) I_4 \cos x_3] - (1+\alpha)\lambda_4 I_4 x_4 \sin x_3 \quad (20)$$

are used to check the existence of singular-arc solutions that $H_{u_1}(t)$ or $H_{u_2}(t)$ stays zero for a period of time. Therefore, if Eq. (19) or (20) is equal to zero when the corresponding switching function (16) or (17) is equal to zero, the singular control is determined by solving

$$\ddot{H}_{u_i}(x,u,\lambda) = 0, \qquad i = 1,2 \quad (21)$$

In addition to the preceding conditions (21), there are transversality conditions

$$\lambda_1(t_f) \cos x_3(t_f)/R + [\lambda_3(t_f) - \lambda_1(t_f)] \cos(\Psi - x_1(t_f) - x_3(t_f)) = 0 \quad (22)$$

$$H = 0 \quad (23)$$

to be satisfied. Note that condition (22) is derived from the special boundary condition (12) for pointing and that condition (23) is derived from the optimality condition of t_f and the property that H is constant along the trajectory.

The two-point boundary-value problem is to find the initial Lagrange multipliers $\lambda_i(0)$, $i = 1,\ldots,4$, and final time t_f to satisfy the Euler-Lagrange equations (5-8) and (14), the switching condition (18) for bang-bang control or Eq. (21) for singular control, the given boundary conditions (10-12), and the transversality conditions (22) and (23). It is noted that accuracy in calculating the switching points is very important since these points implicitly determine the control history. For the guessed initial Lagrange multiplier $\lambda(0)$ and given initial state $x(0)$, the Euler-Lagrange equations (5-8) and (14) are integrated using a variable step-size Runge-Kutta method, and the signs of $H_{u_1}(t)$ and $H_{u_2}(t)$ are checked at every integration step. If the sign of $H_{u_1}(t)$) or $H_{u_2}(t)$ changes, the integration is resumed from the previous point, and the increment of time is adjusted to search the zero point of the switching function using a secant method. The numerical error in calculating switching points must be less than the numerical integration error.

IV. Minimizing-Boundary-Condition Method

In this section, we apply the minimizing-boundary-condition method to solve the two-point boundary-value problem stated in Sec. III.

The minimizing-boundary-condition method was developed in Ref. 10, which used nonlinear optimization techniques to solve two-point boundary-value problems in a way similar to shooting methods. Unlike the shooting methods, it is not sensitive to the first initial guess. Multiple-point shooting methods have been shown to improve the initial guess sensitivities[12]; however, multiple-point shooting methods may lose accuracy in each node point, and it is also very difficult to write a numerical program for multiple-point shooting methods. The minimizing-boundary-condition method has been used successfully for optimal periodic control problems that require a very accurate algorithm since the minimizing periodic solution is extremely close to the steady-state solution and the performance index is very flat.[10] Furthermore, it is easy to write a computer program for the minimizing-boundary-condition method since a large number of existing nonlinear optimization programs can be used.

The minimizing-boundary-condition method is a modified shooting method. A simple shooting method corrects the initial guess by using a linearized function of the initial guess and final boundary condition. Because of the effectiveness of the linearization, the initial guess must be very close to the solution to guarantee a convergence. A modified shooting method was used in Ref. 13 to reduce the number of the initial unknowns by removing one boundary condition and fixing one initial unknown. The solution to the original system is obtained by changing the fixed initial unknown until the removed boundary condition is satisfied. However, this method is still sensitive to the initial guess. The minimizing-boundary-condition method extends the shooting method by removing two boundary conditions without fixing any initial unknowns. An explanation of this method follows.

Two minimization steps are used in the minimizing-boundary-condition method. The first step is to minimize an artificial cost function that is a quadratic form of one of the transversality conditions

$$Q = H^2 \quad (24)$$

with respect to the initial unknowns

$$Z = \begin{bmatrix} \lambda(0) \\ t_f \end{bmatrix} \tag{25}$$

subject to the equality constraints

$$\begin{bmatrix} x_2(t_f) \\ x_4(t_f) \\ \sin x_3(t_f)/R + \sin(\Psi - x_1(t_f) - x_3(t_f)) \end{bmatrix} = 0 \tag{26}$$

where the final state $x(t_f)$ and Lagrange multiplier $\lambda(t_f)$ are obtained by integrating the differential equations (5–8) and (14) with the guessed initial Lagrange multiplier $\lambda(0)$ and given initial state $x(0)$. For the guessed initial values, the switching points, which implicitly determine the control function according to conditions (18), are determined numerically during the integration by calculating the zero point of the switching function using a secant method.

Note that one boundary condition [Eq. (23)] is used to construct the cost function, another boundary condition [Eq. (22)] is eliminated, and the number of initial unknowns is not reduced in this step. Since the boundary conditions are relaxed, the solution of this step is not unique and consists of a family indexed by the Hamiltonian function. The closest family point to the initial guess is most likely to be the solution of this step. Furthermore, since the nearest point to the initial guess is searched, the linearity of the correction algorithm for the minimization program is enhanced. Therefore, the solution of this step converges very quickly. In most cases, the solution of this step can be obtained within eight iterations for a rough initial guess. The second step of the minimizing-boundary-condition method uses the converged solution from the first step as an initial guess to minimize the same artificial cost function Q, subject to the equality constraints of Eq. (26) and the eliminated boundary condition (22). Note that the solution of the second step not only minimizes the artificial cost function Q but also achieves a zero value for Q, which is an equality constraint to be satisfied in the original two-point boundary-value problem.

Solutions for the two-point boundary-value problem of Sec. III converge rapidly for rough initial guesses, and the residual error in boundary condition is within 1.E-8. Since the cost function Q is an implicit function of the switching point, the initial Lagrange multipliers must be selected so that at least one switching point occurs in each control or the cost function Q will not be affected by the change of the initial Lagrange multipliers. Nevertheless, the initial Lagrange multipliers can be obtained very easily by choosing the Lagrange multipliers at the guessed switching point and integrating the differential equation backward to the initial time.

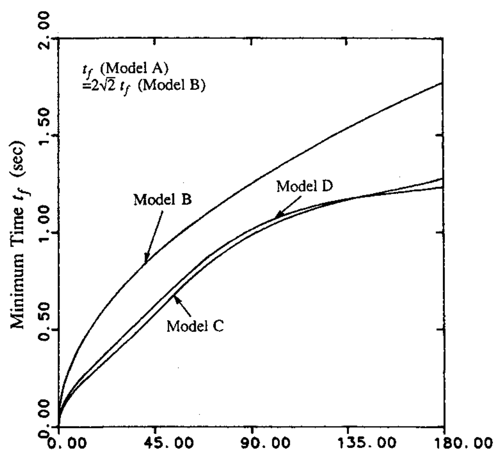

Fig. 3 Minimum times for four models.

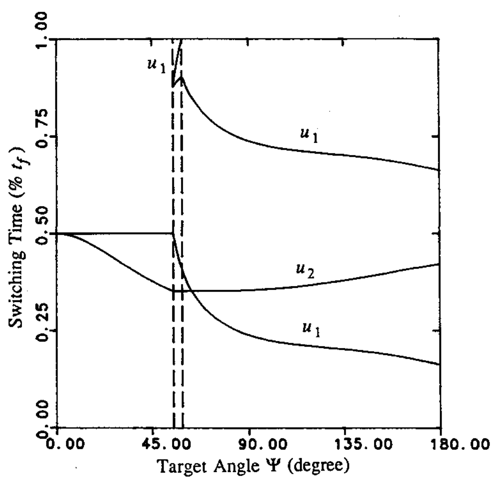

Fig. 4 Switching times for model C.

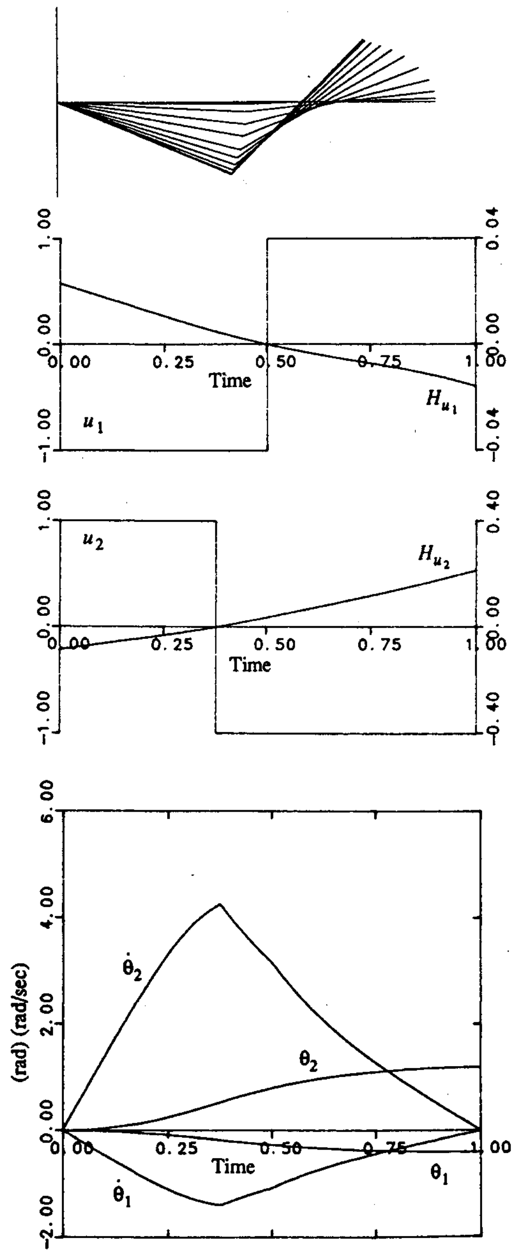

Fig. 5 Solutions for model C (Ψ = 45 deg).

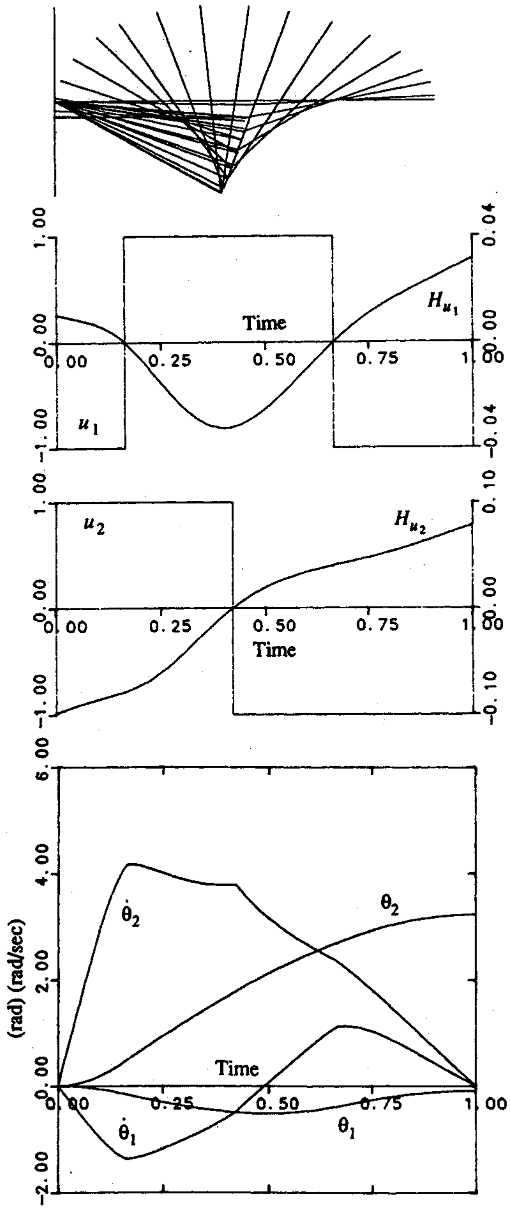

Fig. 6 Solutions for model C (Ψ = 180 deg).

V. Results

The minimum-time optimal control problem of models A and B is similar to that of a double integrator, which can be solved analytically.[14] The minimum-time solutions for models C and D are determined numerically in this section by solving the two-point boundary-value problem with switching conditions using the minimizing-boundary-condition method.

Minimum-Time Solutions for Model A

If $x_3 \equiv 0$, the dynamic equations for model A are reduced to two linear differential equations from Eqs. (5) and (6). The analytical minimum-time solutions are

$$t_f = 2\sqrt{(I_1 + I_2 + m_2L_1^2 + 2m_2rL_1)\Psi} = 2\sqrt{2}\sqrt{\Psi} \tag{27}$$

$$u_1 = \begin{cases} +1 & 0 \le t < t_f/2 \\ -1 & t_f/2 \le t \le t_f \end{cases} \tag{28}$$

$$u_2 = \left(\frac{I_2 + I_4}{I_2 + I_3 + 2I_4}\right)u_1 = (0.3125)u_1 \tag{29}$$

where u_2 of Eq. (29) is the elbow torque required to lock the elbow joint, and $|u_{2_{\max}}| < 1$.

Minimum-Time Solutions for Model B

If $x_1 \equiv 0$, the dynamic equations for model B are reduced to two linear differential equations from (7) and (8). The analytical minimum-time solutions are

$$t_f = 2\sqrt{I_2\Psi} = \sqrt{\Psi} \tag{30}$$

$$u_2 = \begin{cases} +1 & 0 \le t < t_f/2 \\ -1 & t_f/2 \le t \le t_f \end{cases} \tag{31}$$

$$u_1 = \left(1 + \frac{I_4}{I_2}\cos x_3\right)u_2 - I_4x_4^2\sin x_3$$

$$u_1 = (1 + 1.5\cos x_3)u_2 - (0.375)x_4^2\sin x_3 \tag{32}$$

where u_1 of Eq. (32) is the shoulder torque required to lock the shoulder joint, and $|u_{1_{\max}}| > 1$. Figure 3 shows the minimum time t_f of this model vs the target angle Ψ.

Equations (27) and (29) show that the minimum time is proportional to the mass moment of inertia. Therefore, by reducing the mass moment of inertia, the minimum time for model B is reduced to 35.5% of the minimum time for model A. However, the maximum torque of the shoulder control for model B exceeds the torque limit.

Minimum-Time Solutions for Model C

The minimum-time solutions for model C are shown in Fig. 3 for different target angles with a constant target distance (R = 100). It can be seen from Fig. 3 that the minimum times of model C are far shorter than the minimum times of model B for $0 \le \Psi \le 180$ deg. Furthermore, the maximum torques needed for model C are within the limits, whereas model B needs a larger torque to lock the shoulder joint.

The switching time specified by the percentage of t_f vs target angle is shown in Fig. 4. The elbow control u_2 always switches once in one trajectory. For $0 \le \Psi < 56.1$ deg, u_2 switches earlier than u_1. For 56.1 deg $\le \Psi \le$ 180 deg, u_2 switches later than u_1. The switching pattern for the shoulder control u_1 is more complicated. For $0 \le \Psi < 56.1$ deg, u_1 switches only once. For 56.1 deg $\le \Psi <$ 59 deg, u_1 switches three times in one trajectory, and the last two switching points are both very close to t_f. The bang-bang solutions in this region could actually be singular-arc solutions since they might approximate a high-frequency chattering solution. However, the conditions for singular control, Eqs. (19) and (20), are not satisfied at the switching points. For 59 deg $\le \Psi \le$ 180 deg, u_1 switches twice in one trajectory. The overall switching pattern is explained as follows: The elbow control u_2 is used mainly for driving the second link to satisfy the boundary condition in minimum time; therefore, u_2 switches only once. The shoulder control u_1 is used for assisting u_2 to achieve the minimum time; therefore, u_1 requires more complicated switching patterns than u_2.

The control histories and state trajectories are shown in Fig. 5 for Ψ = 45 deg and in Fig. 6 for Ψ = 180 deg. When Ψ = 57.3 deg, the switching function H_{u_1} is very close to zero between the second and third switching points of u_1, as shown in Fig. 7. In Fig. 8, $\dot{H}_{u_1}$ is shown to be not equal to zero at the second and third switching points of u_1. This demonstrates that the optimal solutions are not singular. At the beginning of

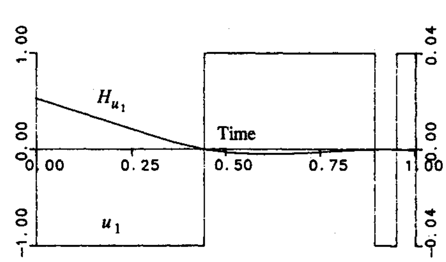

Fig. 7 Switching function H_{u_1} (Ψ = 57.3 deg).

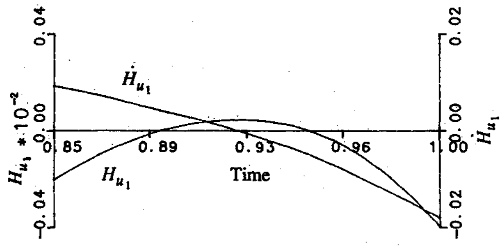

Fig. 8 H_{u_1} and its derivative.

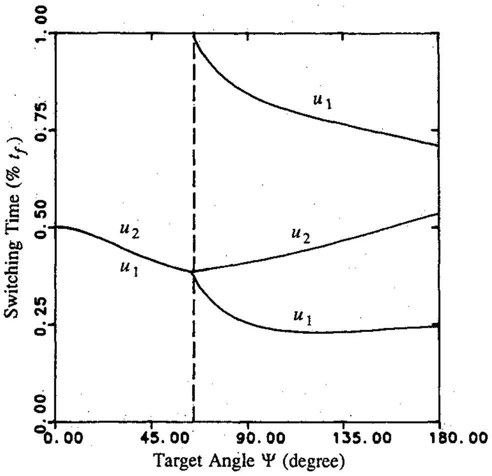

Fig. 9 Switching times for model D.

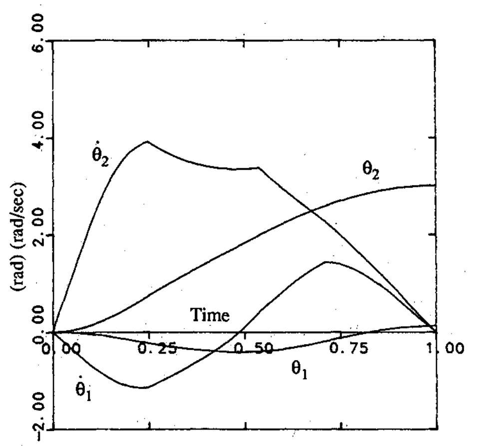

Fig. 10 Trajectories for model D ($\Psi = 180$ deg).

the trajectory, u_1 and u_2 are in opposite directions (u_2 is positive and u_1 is negative). This is also true in Refs. 6-9. The velocities of the two joints reach the maximum magnitude at the first switching point of u_2. For small target angles, the angular velocity of the elbow is positive, and that of the shoulder is negative. For large target angles, the angular velocity of the elbow is still positive, but that of the shoulder is negative or positive. The angular velocity of the shoulder $\dot{\theta}_1$ is almost symmetrical when $\Psi = 180$ deg. The angular rotation of the elbow is always positive, and the shoulder rotation is always negative for $0 \leq \Psi \leq 180$ deg.

Minimum-Time Solutions for Model D

The minimum-time solutions for model D are determined numerically using the minimizing-boundary-condition method, together with a continuation method, which uses the minimum-time solutions for model C by continuously changing the system parameter α from 1 to 0. The minimum times for model D are shown in Fig. 3 for different target angles with a constant target distance. The minimum times for models C and D are very close to each other. This shows that either joint or direct torque applied to the second link is insignificant to the minimum time. However, the trajectories for the two models are different. For example, model D does not have three switching points for u_1 when $56.1 \text{ deg} \leq \Psi < 59$ deg as does model C. Figure 9 shows that the two controls switch at the same time when $0 \leq \Psi < 64.7$ deg, and u_1 switches twice when $64.7 \text{ deg} \leq \Psi \leq 180$ deg. Figure 10 shows that when the target angle $\Psi = 180$ deg, the angular rotation of the shoulder reaches negative values for model D, which is different from model C.

VI. Conclusions

The minimum-time optimal solutions have been determined for four pointing system models in which equal torque limits are assumed for two uniform rigid links. The results show that the pointing system with two degrees of freedom performs much better than the pointing system with one degree of freedom and that either joint or direct torque applied to the system produces almost the same minimum time. Model A is a conventional one-link pointing system. Reducing the mass moment of inertia yields shorter minimum times for model B; however, model B requires a maximum control, which exceeds the torque limit, to lock the shoulder joint. By using extra freedom of the shoulder joint, model C generates shorter minimum times than model B does without violating the torque limits. To achieve a minimum-time maneuver, the shoulder and elbow control proceed in opposite directions at the beginning of the maneuver. The elbow control is mainly responsible for aligning the second link of the manipulator with the target point as quickly as possible, with assistance from the shoulder control. This explains why the switching pattern of the shoulder control is more complicated than that of the elbow control. The performance of model D shows little difference from model C although the trajectories of the two models are different from each other.

The minimizing-boundary-condition method has been applied successfully to the two-point boundary-value problem with two switching functions. The minimum-time optimal solutions converge very quickly for this problem, and the initial guess need not be close to the solution. Furthermore, implementation of the minimizing-boundary-condition method is easier than that of multiple-point shooting methods.

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